

Total number of printed pages-11

3 (Sem-1/CBCS) MAT HC 2

2020

(Held in 2021)

MATHEMATICS

(Honours)

Paper : MAT-HC-1026

(Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : $1 \times 10 = 10$

(a) Find the polar representation of the point $(2, -2)$.

(b) Find the Cartesian co-ordinates of the point $(2, 2\pi/3)$.

(c) If $f : R \rightarrow R$ is given by $f(x) = x^2$, what is $f^{-1}((0, 4))$?

Contd.

- (d) Write the statement and its negation using quantifiers.
 “In each tree in the garden, we can find a branch on which all of the leaves are green”.
- (e) If A is the set of all $n \times n$ symmetric matrices and B is the set of all $n \times n$ real skew-symmetric matrices, what is $A \cap B$?
- (f) Let $M(2, \mathbb{R})$ denote the set of all 2×2 matrices over \mathbb{R} .
 Consider the function $f : M(2, \mathbb{R}) \rightarrow \mathbb{R}$ given by $f(A) = \det A$.
 Show that f is not one-one.
- (g) State the well-ordering principle in \mathbb{N} .
- (h) State ‘true’ **or** ‘false’ with justification :
 If one row in an echelon form of an augmented matrix is $[0 \ 0 \ 0 \ 5 \ 0]$, then the associated linear system is inconsistent.
- (i) State ‘true’ **or** ‘false’ with justification :
 Each column of AB (where A and B are matrices whose product AB is defined) is a linear combination of the columns of B using weights from the corresponding columns of A .

- (j) *Fill in the blanks :*
If A is a triangular matrix then $\det A$ is the product of the entries on the _____.

2. Answer the following questions : $2 \times 5 = 10$

- (a) Compute $(1+i)^{100}$.
- (b) Describe the following set explicitly and mark it on the real line

$$X = \{x \in \mathbb{R} \mid x(x-1)(x-2) < 0\}$$

- (c) Consider the relation on \mathbb{R} defined by $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$. Is this relation an equivalence relation? Justify.

- (d) Find standard matrix of T , where T is a linear transformation such that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotates points about the origin through $\frac{3\pi}{2}$ in a counter-clockwise manner.

- (e) A is 3×3 matrix with three pivot positions. Explain the following—

- (i) Does $A\vec{x} = \vec{0}$ have a nontrivial solution?
- (ii) Does $A\vec{x} = \vec{b}$ have at least one solution for all \vec{b} in \mathbb{R}^3 ?

3. Answer **any four** questions : $5 \times 4 = 20$

(a) If $n|q$ then prove that any root of $z^n - 1 = 0$ is a root of $z^q - 1 = 0$. Prove that the common roots of $z^m - 1 = 0$ and $z^n - 1 = 0$ are roots of $z^d - 1 = 0$ where $d = \text{g.c.d}(m, n)$

i.e. $U_m \cap U_n = U_d$. 1+4=5

(b) Let $X = \mathbb{R} = Y$. Let $A = \{1\}$ and $B = \mathbb{R}$. Draw the sketch of $A \times B$ as a subset of \mathbb{R}^2 .

For $A \subseteq X$ and $B \subseteq Y$ show that there may be subsets of $X \times Y$ that are not of the form $A \times B$. 2+3=5

(c) For any sets A and B , show that the following are equivalent. 5

(i) $A \subseteq B$

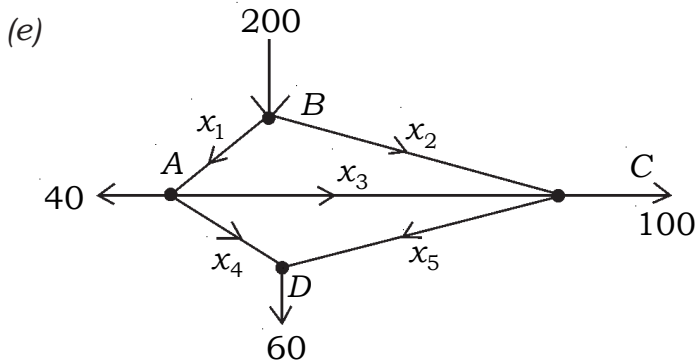
(ii) $A \cup B = B$

(iii) $A \cap B = A$

(iv) $B^c \subseteq A^c$

- (d) Describe the solutions of the following system in parametric form. 5

$$\begin{aligned}x_1 + 3x_2 - 5x_3 &= 4 \\x_1 + 4x_2 - 8x_3 &= 7 \\-3x_1 - 7x_2 + 9x_3 &= -6\end{aligned}$$



For the figure above find the general traffic pattern in the freeway network (Flow rates are cars/minute)

Describe the general traffic pattern when the road whose flow is x_4 is closed. 5

- (f) Use Cramer's Rule to compute the solutions of the system 5

$$\begin{aligned}2x_1 + x_2 + x_3 &= 4 \\-x_1 + \quad + 2x_3 &= 2 \\3x_1 + x_2 + 3x_3 &= -2\end{aligned}$$

4. Answer **any four** of the following :
10×4=40

(a) (i) Compute $z^n + \frac{1}{z^n}$ if $z + \frac{1}{z} = \sqrt{3}$.
5

(ii) Prove that 5

$$\begin{aligned} \sin 5t &= 16 \sin^5 t - 20 \sin^3 t + 5 \sin t \\ \cos 5t &= 16 \cos^5 t - 20 \cos^3 t + 5 \cos t \end{aligned}$$

(b) (i) Show that the function
 $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by
 $f(x, y) = (ax + by, cx + dy)$
is a bijection of $ad - bc \neq 0$.
Find the inverse of f . 5

(ii) For any sets A, B, C prove that
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ 5

(c) (i) Show that a function $f : X \rightarrow Y$
is one-one if and only if
 $f(A \cap B) = f(A) \cap f(B)$ holds for
all subsets A and B of X . 6

(ii) Give an example of a relation that
is reflexive and transitive but not
symmetric. 2

(iii) Produce a counter example to disapprove the statement —
“For integers a, b, c if a divides bc , then a divides b or a divides c .”
2

(d) Prove that if a, b are integers not both zero and d be the greatest common divisor of a and b then \exists integers x, y s.t. $d = ax + by$. Further, prove that two integers m and n are relatively prime if and only if \exists integers p and q s.t. $pm + qn = 1$.
6+4

(e) (i) Prove the statement using contrapositive
“For integers x, y if $x + y$ is even, then x and y are both odd or both even”.
2

(ii) Determine h such that the following matrix is the augmented matrix of a consistent linear system

$$\begin{bmatrix} 2 & 8 & h \\ 4 & 6 & 7 \end{bmatrix} \quad 2$$

- (iii) Determine if \vec{b} is a linear combination of the vectors formed from the column of A . 4

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

- (iv) Show that the set of two vectors $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. 2

- (f) (i) If A is a $m \times n$ matrix and \vec{U}, \vec{V} are vectors in \mathbb{R}^n , C is a scalar then prove that

$$A(\vec{U} + \vec{V}) = A\vec{U} + A\vec{V}$$

$$\text{and } A(C\vec{U}) = C(A\vec{U}) \quad 4$$

- (ii) If A is a square $n \times n$ matrix then prove that the following statements are logically equivalent :

(a) A is an invertible matrix

(b) There is an $n \times n$ matrix C such that $CA = I$.

(c) The equation $A\vec{x} = \vec{0}$ has only the trivial solution.

- (d) A has n pivot positions
 (e) A is row equivalent to the $n \times n$ identity matrix. 6

- (g) (i) Write the following system in matrix form and use the inverse of the co-efficient matrix to solve

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7 \end{aligned} \quad 2$$

- (ii) Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T .

Then show that T is invertible if and only if A is an invertible matrix. Show that the linear transformation S given by $S(\vec{x}) = A^{-1}\vec{x}$ where $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$, is the unique inverse of T . 4

- (iii) Find the inverse of the following matrix if it exists by performing suitable row operations on the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & \\ -3 & 1 & 4 & \\ 2 & -3 & 4 & \end{array} \right] \quad 4$$

- (h) (i) Compute the determinant by co-factor expansion choosing the row or column that involves least amount of computation

$$\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix} \quad 2$$

- (ii) State 'true' **or** 'false' with justification :

The $(i, j)^{th}$ cofactor of a matrix A is the matrix A_{ij} obtained from A by deleting the i^{th} row and j^{th} column of A . 2

- (iii) For the matrix given below

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

- (a) compute the determinant.
- (b) what is determinant of an elementary row replacement of the matrix?
- (c) what is the determinant of an elementary scaling matrix with k on the diagonal? 3

(iv) Use a determinant to decide if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, where

$$\vec{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

3
