3 (Sem–2/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–2016

(Real Analysis)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** questions : 1×10=10

(a) Find the infimum of the set

$$\left\{1-\frac{(-1)^n}{n} : n \in N\right\}$$

(b) If A and B are two bounded subsets of $\mathbb{R}$, then which one of the following is true ?

(i) $sup(A \cup B) = sup\{sup A, sup B\}$

(ii) $sup(A \cup B) = sup A + sup B$

(iii) $sup(A \cup B) = sup\ A \,.\, sup\ B$

(iv) $sup(A \cup B) = sup\ A \cup sup\ B$

(c) There does not exist a rational number x such that $x^2 = 2$. *(Write True **or** False)*

(d) The set Q of rational numbers is uncountable. *(Write True **or** False)*

(e) If $I_n = \left(0, \frac{1}{n}\right)$ for $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} I_n = ?$

(f) The convergence of $\{|x_n|\}$ imply the convergence of $\{x_n\}$.

*(Write True **or** False)*

(g) What are the limit points of the sequence $\{x_n\}$, where $x_n = 2 + (-1)^n$, $n \in \mathbb{N}$?

(h) If $\{x_n\}$ is an unbounded sequence, then there exists a properly divergent subsequence. *(Write True **or** False)*

(i) A convergent sequence of real numbers is a Cauchy sequence.

*(Write True **or** False)*

(j) If $0 < a < 1$ then $\lim_{n\to\infty} a^n = ?$

(k) The positive term series $\sum \frac{1}{n^p}$ is convergent if and only if

(i) $p > 0$

(ii) $p > 1$

(iii) $0 < p < 1$

(iv) $p \le 1$

(Write correct one)

(l) Define conditionally convergent of a series.

(m) If $\{x_n\}$ is a convergent monotone sequence and the series $\sum_{n=1}^{\infty} y_n$ is convergent, then the series $\sum_{n=1}^{\infty} x_n y_n$ is also convergent.

*(Write True **or** False)*

(n) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^m\left(1+\frac{1}{n^p}\right)}$

where m and p are real numbers under which of the following conditions does the above series convergent ?

(i) $m > 1$

(ii) $0 < m < 1$ and $p > 1$

(iii) $0 \leq m \leq 1$ and $0 \leq p \leq 1$

(iv) $m = 1$ and $p > 1$

(o) Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers defined by $x_1 = 1$, $y_1 = \frac{1}{2}$,

$x_{n+1} = \frac{x_n + y_n}{2}$ and $y_{n+1} = \sqrt{x_n y_n}$ $\forall n \in \mathbb{N}$

then which one of the following is true ?

(i) $\{x_n\}$ is convergent, but $\{y_n\}$ is not convergent

(ii) $\{x_n\}$ is not convergent, but $\{y_n\}$ is convergent

(iii) Both $\{x_n\}$ and $\{y_n\}$ are convergent and $\lim_{n\to\infty} x_n > \lim_{n\to\infty} y_n$

(iv) Both $\{x_n\}$ and $\{y_n\}$ are convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$

2. Answer ***any five*** parts : 2×5=10

(a) If a and b are real numbers and if $a < b$, then show that $a < \frac{1}{2}(a+b) < b$.

(b) Show that the sequence $\left\{\frac{2n-7}{3n+2}\right\}$ is bounded.

(c) If $\{x_n\}$ converges in $\mathbb{R}$, then show that $\lim_{n\to\infty} x_n = 0$

(d) Show that the series 1+2+3+...., is not convergent.

(e) Test the convergence of the series :

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...$$

(f) State Cauchy's integral test of convergence.

(g) State the completeness property of $\mathbb{R}$ and find the $sup\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

(h) Does the Nested Interval theorem hold for open intervals ? Justify with a counter example.

3. Answer ***any four*** parts : 5×4=20

(a) If x and y are real numbers with $x < y$, then prove that there exists a rational number r such that $x < r < y$.

(b) Show that a convergent sequence of real numbers is bounded.

(c) Prove that $\lim_{n \to \infty}\left(n^{\frac{1}{n}}\right) = 1$.

(d) $\{x_n\}$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Show that the sequence $\{\sqrt{x_n}\}$ of positive square roots converges and $\lim_{n \to \infty} \sqrt{x_n} = \sqrt{x}$.

(e) Show that every absolutely convergent series is convergent. Is the converse true ? Justify. 4+1=5

(f) Using comparison test, show that the series $\sum\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$ is convergent.

(g) State Cauchy's root test. Using it, test the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots\ldots$$

1+4=5

(h) Show that the sequence defined by the recursion formula

$$u_{n+1}=\sqrt{3u_n},\ u_1=1$$

is monotonically increasing and bounded. Is the sequence convergent ? 2+2+1=5

4. Answer ***any four*** parts : 10×4=40

(a) Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is convergent and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ which lies between 2 and 3.

(b) *(i)* Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences of real numbers such that $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$.

Show that $\{y_n\}$ is convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$ 5

(ii) What is an alternating series ? State Leibnitz's test for alternating series. Prove that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\infty$ is a conditionally convergent series. 1+1+3=5

(c) Test the convergence of the series

$$1+a+a^2+\ldots\ldots+a^n+\ldots\ldots$$

(d) (i) Using Cauchy's condensation test, discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ 5

(ii) Define Cauchy sequence of real numbers. Show that the sequence $\left\{\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots+\frac{1}{n!}\right\}$ is a Cauchy sequence. 1+4=5

(e) (i) Show that a convergent sequence of real numbers is a Cauchy sequence. 5

(ii) Using Cauchy's general principle of convergence, show that the sequence $\left\{1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}\right\}$ is not convergent. 5

(f) (i) Prove that every monotonically increasing sequence which is bounded above converges to its least upper bound. 5

(ii) Show that the limit if exists of a convergent sequence is unique. 5

(g) State and prove p-series.

(h) (i) Test the convergence of the series

$$x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \ldots\ldots + \frac{n^2 - 1}{n^2 + 1}x^n + \ldots\ldots (x > 0)$$

5

(ii) If $\{x_n\}$ is a bounded increasing sequence then show that $\lim_{n \to \infty} x_n = \sup\{x_n\}$ 5

(i) (i) Show that a bounded sequence of real numbers has a convergent subsequence. 5

(ii) State and prove Nested Interval theorem. 5

(j) (i) Show that Cauchy sequence of real numbers is bounded. 5

(ii) Test the convergence of the series

$$x^2 + \frac{2^2}{3.4}x^4 + \frac{2^2.4^2}{3.4.5.6}x^6 + \frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8 + \ldots\ldots (x > 0)$$

5

3000